Math 279 Lecture 22 Notes

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1 Applying Regularity Structures to Rough Path Theory and Singular PDEs

1.1 Recovering a previous theorem as an application of the reconstruction theorem

For our rough path theory, we choose $A = \{\alpha - 1, 2\alpha - 1, 0, \alpha\}$ with $\alpha \in (1/3, 1/2)$. Here, $T_\alpha = \langle X_1, \dots, X_\ell \rangle$, where we think of $X = (X_1, \dots, X_\ell)$ as an abstract candidate for the path $x(\cdot) \in \mathcal{C}^\alpha$, $T_{\alpha-1} = \langle \dot{X}_1, \dots, \dot{X}_\ell \rangle$, and $T_{2\alpha-1} = \langle \dot{\mathbb{X}}^{i,j} : 1 \leq i, j \leq \ell$. We think of $\dot{\mathbb{X}} = [\mathbb{X}^{i,j}] = X \otimes \dot{X}$. From this, we have

$$\Gamma_h X = X + h\mathbf{1}, \qquad \Gamma_h \dot{X} = \dot{X}, \qquad \Gamma_h (X \otimes \dot{X}) = X \otimes \dot{X} + h \otimes \dot{X}.$$

Recall that $f : \mathbb{R}^d \to \bigoplus_{\alpha < \gamma} T_\alpha \in \mathcal{C}_M^\gamma$ means $\|f(s) - \Gamma_{st} f(t)\|_\alpha \lesssim |s - t|^{\gamma - \alpha}$. So if we decrease the index, the regularity required would be rougher. Last time, we argued that if $Y(t) = y(t)\mathbf{1} + \widehat{y}(t) \cdot X \in \mathcal{C}_M^{2\alpha}$, then the pair $\mathbf{y}(t) = (y(t), \widehat{y}(t)) \in \mathscr{G}^\alpha(x)$, i.e.

$$|\widehat{y}(t) - \widehat{y}(s)| \lesssim |t - s|^\alpha, \qquad |y(t) - y(s) - \widehat{y}(s) x(s,t)| \lesssim |t - s|^{2\alpha}.$$

Now we want to examine another algebraic manipulation in our abstract setting, namely we wish to make sense of $Y \cdot \dot{X}$, which we want to think of as $(y\mathbf{1} + \widehat{y}X) \cdot \dot{X} = y\dot{X} + \widehat{X} \otimes \dot{X}$. Because of this, consider

$$(Y \cdot \dot{X})(t) = y\dot{X} + \widehat{y}\dot{\mathbb{X}}.$$

Proposition 1.1. *$(y, \widehat{y}) \in \mathscr{G}^\alpha(x)$ if and only if $Y \cdot \dot{X} \in \mathcal{C}_M^{3\alpha - 1}$.*

Proof.

$$\begin{aligned}(Y \cdot \dot{X})(s) - \Gamma_{s,t}(Y \cdot \dot{X})(t) &= (y(s)\dot{X} + \widehat{y}(s)\dot{\mathbb{X}}) - (y(t)\dot{X} + \widehat{y}(t)\dot{\mathbb{X}} + \widehat{(}s)x(t,s)\dot{X}) \\ &= (y(s) - y(t) - \widehat{y}(s)x(t,s))\dot{X} + (\widehat{y}(s) - \widehat{y}(t))\dot{\mathbb{X}}\end{aligned}$$

For the first coefficient, we want the estimate

$$|y(s) - y(t) - \widehat{y}(s)x(t - s)| \lesssim |t - s|^{\gamma - (\alpha - 1)} = |t - s|^{2\alpha}.$$

This is exactly the estimate for the Gubinelli derivative. Similarly, we want

$$|\widehat{y}(s) - \widehat{y}(t)| \lesssim |t-s|^{\gamma-(2\alpha-1)} = |t-s|^{\alpha}.$$

This gives the equivalence. $\square$

Now we wish to apply our reconstruction theorem to $Y \cdot \dot{X}$. More precisely, there exists some operator $J_M^{3\alpha-1}$ on $\mathcal{C}_M^{3\alpha-1}$ such that $W := J_M^{3\alpha-1}(Y \cdot \dot{X})$ satisfies

$$|(W - \Pi_t(y(t)\dot{X} + +\widehat{y}(t)\dot{\mathbb{X}}))(\psi_t^{\delta})| \lesssim \delta^{3\alpha-1}.$$

Equivalently,

$$|W(\psi_t^{\delta}) - (y(t)\dot{x} + \widehat{y}(t)\mathbb{X}_t(t,\cdot))(\psi_t^{\delta})| \lesssim \delta^{3\alpha-1}.$$

This is indeed the first theorem we proved in this class, namely given $\mathbf{y} = (y, \widehat{y}) \in \mathscr{G}^{\alpha}(x)$ and $\mathbf{x} = (x, \mathbb{X}) \in \mathscr{R}_{\alpha,2\alpha}$, there exists $z \in \mathcal{C}^{\alpha}$ such that

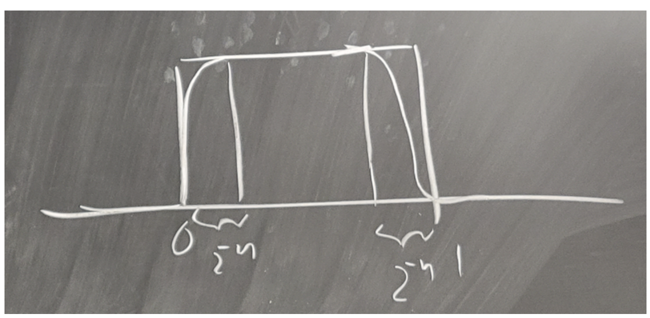
$$|z(s) - z(t) - y(t)(x(s) - x(t)) - \widehat{y}(t)\mathbb{X}(t,s)| \lesssim |t-s|^{3\alpha-1}$$

with $\dot{z} = W$.

To derive this theorem from estimate above it, we need to allow a ψ that is of the form $\psi(t) = \mathbb{1}_{[0,1]}(t)$ sot that $\psi_t^{\delta}(s) = \frac{1}{\delta}\mathbb{1}_{[t,t+\delta]}(s)$. This can be achieved by writing

$$\mathbb{1}_{[0,1]} = \sum_{n=0}^{\infty} \varphi_n(t) + \psi_n(t),$$

where ψ_n, φ_n are smooth with compact support, $\operatorname{supp} \varphi_n \subseteq [0, 2^{-n}]$, and $\operatorname{supp} \psi_n \subseteq [1 - 2^{-n}, 1]$.



1.2 Applying regularity structure theory to understand a singular PDE

We now turn our attention to one of our singular PDE, say the KPZ equation

$$\begin{cases} h_t = h_{xx} + h_x^2 + \xi - C \\ h(x,0) = h^0(x), \end{cases}$$

where ξ is white noise. As we argued before, if $\xi^\varepsilon = \xi *_x \chi^\varepsilon$ with $\chi^\varepsilon(x) = \frac{1}{\varepsilon}\chi(\frac{x}{\varepsilon})$, then the corresponding PDE

$$h_t^\varepsilon = h_{xx}^\varepsilon + (h_x^\varepsilon)^2 + \xi^\varepsilon - C_\varepsilon$$

is well-posed, and $\lim_{\varepsilon\to 0} h^\varepsilon$ exists only if $C_\varepsilon \approx C/\varepsilon$, where $C = \frac{1}{2}\int \chi^2$ (a theorem due to Martin Hairer).

To achieve this, we first build an abstract version of our PDE and usr it to have an abstract solution that is continuous with respect to its input (which in cludes a well-selected version of ξ). Indeed, if we write $\mathcal{P}$ for the operator/kernel $(\partial_t - \partial_x^2)^{-1}$, then

$$h = \mathcal{P} * (h_x^2 + \xi - C) + \mathcal{P} * h^0 = \mathcal{F}(h).$$

Then we would make sense of $\mathcal{F}$ in a suitable way, show that $\mathcal{F}$ has a fixed point, and this would be our candidate for the solution. For this, we need some preparations.

Definition 1.1. Given a regularity structure (A, T, G), we say $V \subseteq T$ is a **sector** if $V = \bigoplus_{\alpha\in A} V_\alpha$ with subspaces $V_\alpha \subseteq T_\alpha$ and $G(V) \subseteq V$.

Definition 1.2. If $\mathcal{L} = \sum_{|k|=r} a_k \partial^k$ is a differential operator, we say $\widehat{\mathcal{L}} : V \to T$ **represents** $\mathcal{L}$ if the following conditions hold:

- If $\tau \in V_\alpha$, then $\widehat{\mathcal{L}}\tau \in T_{\alpha - r}$.
- $\widehat{\mathcal{L}}\Gamma_h = \Gamma_h\widehat{\mathcal{L}}$.
- $\Pi_a\widehat{\mathcal{L}}\tau = \mathcal{L}(\Pi_a\tau)$.

We can also talk about products. In other words, we want to be able to multiply $f \in \mathcal{C}_M^\alpha$ and $g \in \mathcal{C}_M^\beta$ to get $f \odot g \in \mathcal{C}_M^{\alpha\wedge\beta}$.

Recall that i we have a distribution F, then we can talk about

$$F * K \text{ ``=''} \int F(y)K(x-y)\,dy = F(y)\widetilde{K}(y-x)\,dy = \int F(y)\widetilde{K}_x(y)\,dy,$$

where $\widetilde{K}(y) = K(-y)$, which suggests that we should define

$$(F * K)(\varphi) := F(\widetilde{K} * \varphi).$$

For our purposes, we need to examine the regularity of $F * K$. A Schauder-type estimate allows us to show that if K is singular at 0 with singularity of the form $|x|^{\alpha-d}$, then

$$F \in \mathcal{C}^\gamma \implies F * K \in \mathcal{C}^{\gamma+\alpha}.$$

Here is the precise statement:

Theorem 1.1. *Assume that $K : \mathbb{R}^d \to \mathbb{R}$ with the following conditions:*

1. $\operatorname{supp} K \subseteq B_1(0)$

2. $K \in \mathcal{C}^\infty$ *(this can be relaxed), and* $|\partial^\ell K(x)| \le c_\ell |x|^{\alpha-d-|\ell|}$ *for all* x*.*

Then

$$F \in \mathcal{C}^\gamma \implies F * K \in \mathcal{C}^{\gamma+\alpha}$$

for all $\gamma \in \mathbb{R}$*, though for* $\gamma \in \mathbb{Z}$*, we need to replace the Hölder spaces with Hölder-Zygmund spaces.*

For the proof, we need a suitable candidate for function spaces that are equivalent to Hölder spaces (and its variant would yield Besov spaces), except when $\gamma \in \mathbb{Z}$. For $\gamma < 0$, we have already discussed this; if r is the smallest integer such that $r + \gamma \ge 0$, then define

$$[u]_{\gamma,K} = \sup_{x\in K} \sup_{\alpha\in(0,1]} \sup_{\varphi\in\mathcal{D}_r} \frac{|u(\varphi_x^\delta)|}{\delta^\gamma}$$

where $\mathcal{D}_r = \{\varphi : \|\varphi\|_{C^r} \le 1, \operatorname{supp}\varphi \subseteq B_1(0)\}$, and let

$$\mathcal{C}^\gamma_{\text{loc}} = \{u : [u]_{\gamma,K} < \infty \text{ for all compact } K\}.$$

As for $\gamma > 0$ with $\gamma = n + \gamma_0$ and $n \in \mathbb{N}$, define

$$[u]_{\gamma,K} = \sup_{x\in K} \sup_{\alpha\in(0,1]} \sup_{\varphi\in\mathcal{D}^n} \frac{|\langle u, \varphi_x^\delta\rangle|}{\delta^\gamma},$$

where $\mathcal{D}^n$ is the set of $\varphi \in \mathcal{D}$ such that $\int \varphi P(x)\,dx = 0$ for all polynomials P with $\deg P \le n$. It requires proof to show that when $\gamma \ne \mathbb{Z}$, then these equivalent to the Hölder norms.